to be published in JMP in August, 2006

Contraction of broken symmetries via Kac-Moody formalism

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February 7, 2008

Abstract

I investigate contractions via Kac-Moody formalism. In particular, I show how the symmetry algebra of the standard 2-D Kepler system, which was identified by Daboul and Slodowy as an infinite-dimensional Kac-Moody loop algebra, and was denoted by $\mathbb{H}_2$, gets reduced by the symmetry breaking term, defined by the Hamiltonian

$$H(\beta) = \frac{1}{2m}(p_1^2 + p_2^2) - \frac{\alpha}{r} - \beta\, r^{-1/2} \cos((\varphi - \gamma)/2)\ .$$

For this $H(\beta)$ I define two symmetry loop algebras $\mathfrak{L}_i(\beta),\ i = 1, 2$, by choosing the 'basic generators' differently. These $\mathfrak{L}_i(\beta)$ can be mapped isomorphically onto subalgebras of $\mathbb{H}_2$, of codimension 2 or 3, revealing the reduction of symmetry. Both factor algebras $\mathfrak{L}_i(\beta)/I_i(E, \beta)$, relative to the corresponding energy-dependent ideals $I_i(E, \beta)$, are isomorphic to $\mathfrak{so}(3)$ and $\mathfrak{so}(2, 1)$ for $E < 0$ and $E > 0$, respectively, just as for the pure Kepler case. However, they yield two different non-standard contractions as $E \to 0$, namely to the Heisenberg-Weyl algebra $\mathfrak{h}_3 = \mathfrak{w}_1$ or to an abelian Lie algebra, instead of the Euclidean algebra $\mathfrak{e}(2)$ for the pure Kepler case. The above example suggests a general procedure for defining generalized contractions, and also illustrates the *'deformation contraction hysteresis'*, where contraction which involve two contraction parameters can yield different contracted algebras, if the limits are carried out in different order.

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1 Introduction

In 1926 Pauli [1] obtained the energy levels of the relativistic hydrogen atom algebraically, by using the conserved angular-momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ and the Hermitian form of the Laplace-Runge-Lenz vector

$$\boldsymbol{A} = \frac{1}{2}[\boldsymbol{p} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{p}] - m\alpha\hat{\boldsymbol{r}} \tag{1}$$

The commutation relations among their components are given by

$$\begin{aligned} [L_i, L_j] &= i\hbar\, \epsilon_{ijk} L_k \ , \qquad i, j, k = 1, 2, 3 \\ [L_i, A_j] &= i\hbar\, \epsilon_{ijk} A_k \ , \\ [A_i, A_j] &= -i2mH\ \hbar\, \epsilon_{ijk} L_k \ , \end{aligned} \tag{2}$$

where H is the Hamiltonian of the non-relativistic 3D hydrogen atom. The commutation relations in (2) do not define a closed algebra, since the H on the rhs of (2) is an *operator* and not a number. To obtain nevertheless closed algebras physicists for seventy years have replaced the Hamiltonian H by its eigenvalues E, and thus obtained three different identifications of the symmetry algebra of the hydrogen atom, namely $\mathfrak{so}(4), \mathfrak{so}(3,1)$ and $\mathfrak{e}(3)$, for $E < 0, E > 0$ and $E = 0$, respectively [2]. The same conclusion can be reached by formally 'normalizing' the Runge-Lenz vector $\boldsymbol{A}$ by dividing it by $\sqrt{2m|H|}$, but the resulting quotient vector becomes infinite for $H = 0$.

Instead of the above 'conventional procedure' , Daboul and Slodowy [3] showed that one can obtain a single closed algebra based on the commutation relations (2). This algebra is spanned by the following infinite set of generators

$$\mathbb{H}_3 := \{h^n L_i, h^n A_i \mid i = 1, 2, 3,\ n = 0, 1, \ldots\} \ , \qquad \text{where} \quad h := -2mH \ . \tag{3}$$

The algebra $\mathbb{H}_3$ and its generalizations $\mathbb{H}_N$, the symmetry algebras of the N-dimensional hydrogen atom, were identified [3, 4] as positive loop algebras of twisted or untwisted Kac-Moody algebras [5, 6], for N odd or even, respectively. They were called the *hydrogen algebras*. The above formalism will be reviewed in Sec. 2, and applied to $\mathbb{H}_2$, the hydrogen algebra of the standard 2D Kepler system, defined by the Hamiltonian H_0 of Eq. (5) below.

The algebras $\mathbb{H}_N$ depend on the Hamiltonian H, but *not* on its energy eigenvalues E. However, one can reproduce the usual three corresponding finite-dimensional algebras, $\mathfrak{so}(N+1), \mathfrak{so}(N,1)$ and $\mathfrak{e}(N)$, as factor algebras

$\mathbb{H}_N/I(E)$ relative to energy-dependent ideals $I(E)$; The ideals and factor-algebra formalism will be discussed and applied to $\mathbb{H}_2$ in Sec. 2.1.

In the present paper I investigate *what happens to the algebra* $\mathbb{H}_2$ *and its factor algebra, if the original symmetry of the 2D hydrogen atom is broken.* In particular, I shall study the following Hamiltonian

$$H := H_0 - \beta r^{-1/2} \cos\left[\frac{1}{2}(\varphi - \gamma)\right] \ , \qquad (\gamma = 0 \text{ in the present paper}) \tag{4}$$

where H_0 is the Hamiltonian of the 2-dimensional Kepler problem

$$H_0 := \frac{1}{2m}(p_1^2 + p_2^2) - \frac{\alpha}{r} = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - \frac{\alpha}{r} \ . \tag{5}$$

Throughout this paper I shall set the phase angle γ in (4) equal to zero, since it can always be removed by appropriate choice of the coordinate system (See however the discussion in section 5 below).

The Hamiltonian (4) has an interesting history: It was discovered by Winternitz *et. al.* [7] already in 1967 in their systematic search for super-integrable systems. It was also derived in a more general complex form by T. Sen [8, Eq. (3.14)] in 1987.
The symmetry of (4) was originally studied by Gorringe and Leach [9] in 1993 and recently reviewed by Leach and Flessas [10, §3.3] (see also [11]). The above authors followed the conventional method and found that the symmetry algebras of (4) are $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ for $E < 0$ and $E > 0$, exactly as for the pure 2D Kepler problem (5). However, for $E = 0$ they obtained the Heisenberg-Weyl algebra $\mathfrak{h}_3 = \mathfrak{w}_1$ (which they denoted by $W(3,1)$) [9, 10], instead of the Euclidean algebra $\mathfrak{e}(2)$ for the Kepler case (5).

This result was intriguing, since the symmetry breaking does not affect the symmetry for $E \neq 0$, and only affect it for $E = 0$. And I wondered whether and how the above type of symmetry breaking can be treated via the Kac-Moody formalism. It turned out, that the symmetry algebra of (4) can be treated, via the Kac-Moody formalism, similar to the pure Kepler case, with some important modifications. For example, it is possible to describe the symmetry algebra of (4) by two loop algebras, $\mathfrak{L}_1$ and $\mathfrak{L}_2$, depending on the choice of the *'basic generators'*. It is remarkable that these two algebras can be mapped onto subalgebras of $\mathbb{H}_2$ of codimension 2 and 3, *i.e.* $\mathbb{H}_2$ is larger than these image subalgebras by only 2 and 3 generators, out of infinitely many. The 'missing' generators are manifestations of the symmetry breaking.

Moreover, I will show that the factor algebras $\mathfrak{L}_i/I_i(E,\beta)$ relative to the corresponding energy-dependent ideals yield different types of *contractions* [12, 13, 14], which are included in table 1. This result is important, since the contraction procedure for the above specific system can be generalized to other algebras, as discussed in the summary section.

In Sec. 2 I review Inönü-Wigner contraction and its generalization and in Sec. 3 I review the construction of the hydrogen algebra $\mathbb{H}_2$ for the pure 2-dimensional Kepler problem (5). In sections 4 and 5 I construct two loop algebras $\mathfrak{L}_1$ and $\mathfrak{L}_2$ for the system (4) and their factor algebras $\mathfrak{L}_i/I_i(E,\beta)$. In Sec. 6 I map the $\mathfrak{L}_i$ onto subalgebras $\mathbb{H}_2$, and as I already noted, I shall suggest a general procedure for defining contraction via Kac-Moody formalism and then give some conclusions.

2 Review of generalized Inönü-Wigner contraction

There are many formulations of contractions [12, 13]. I shall give my own definition and notation:

Definition. Let $\mathfrak{g} := \langle X_a, C_{ab}^c \rangle$ be a finite-dimensional Lie algebra with a basis $X_a, a = 1, 2, \ldots, N$ and structure constants C_{ab}^c, and let the parameter-dependent Lie algebra $\mathfrak{g}^\epsilon := \langle X_a^\epsilon, C_{ab}^c(\epsilon) \rangle$ be defined, such that the one-to-one linear map f_ϵ between $\mathfrak{g}$ and $\mathfrak{g}^\epsilon$,

$$f_\epsilon : \mathfrak{g} \mapsto \mathfrak{g}^\epsilon, \qquad f_\epsilon(X_a) = \epsilon^{-n_a} \, X_a^\epsilon \tag{6}$$

is an isomorphism of Lie algebras as long as $\epsilon \neq 0$. If the powers n_a satisfy the condition,

$$n_a + n_b \geq n_c \tag{7}$$

then the limit algebra $\mathfrak{g}^0 = \langle X_a^0, C_{ab}^c(0) \rangle$ with the structure constants

$$C_{ab}^c(0) := \lim_{\epsilon \to 0} C_{ab}^c(\epsilon)$$

exists and it is called the *contracted algebra*. I shall refer to $\mathfrak{g}^\epsilon$ as the *contracting algebra* and to its generators X_a^ϵ as the *contracting generators*.

It is important to emphasize that X_a^ϵ *and* X_a^0 *denote the generators of the Lie algebras* $\mathfrak{g}^\epsilon$ *and* $\mathfrak{g}^0$ *which are defined via the structure constants* $C_{ab}^c(\epsilon)$ *and* $C_{ab}^c(0)$*, respectively.* Therefore, the X_a^0 *are NOT to be regarded as the limits*

of X_a^ϵ *for* $\epsilon \to 0$. Thus, the X_a^0 *always* exist, by definition, as generators of the contracted algebra $\mathfrak{g}^0$, even though representations $r(X_a^\epsilon)$ of $\mathfrak{g}^\epsilon$ might exist with some of the generators having vanishing limits, *i.e.* $\lim_{\epsilon\to 0} r(X_b^\epsilon) = 0$. Such representations could be called *not saved or un-saved*. Otherwise, they are called *saved representations* [15]. Actually in section 5 I shall give a realization of a saved representation of an algebra whose contraction yields an abelian algebra, *i.e.* $C_{ab}^c = 0$ for all a, b, c. For this contraction even the adjoint representation [13] is not saved.

Usually X_a is used also to denote the contracting and contracted generators X_a^ϵ and X_a^0 [13]. This convention is probably used to avoid confusing X_a^0 as the limit of X_a^ϵ for $\epsilon \to 0$. To distinguish the algebras $\mathfrak{g}^\epsilon$ from $\mathfrak{g}$ and $\mathfrak{g}^0$ one attaches an index ϵ to the commutators $[,]_\epsilon$, as it is done in (8) below. I find this usual notation confusing, since X_a^0 are the generators of a different algebra $\mathfrak{g}^0$. I prefer attaching the ϵ to the generators but keep the commutator symbol $[,]$ unchanged. This notation is more useful and user-friendly, especially if matrix representations exist, since one uses $[A, B] = AB - BA$ and the standard matrix multiplication, whether the matrices A and B represent generators of the original or the contracted algebras.

In contrast to the formal definition of X_a^0, the limits $r(X_b^0) := \lim_{\epsilon\to 0} r(X_b^\epsilon)$ of *representations or realizations* $r(X_a^\epsilon)$, if they exist, should satisfy the commutation relations of $\mathfrak{g}^0$, although some of these representations may not be saved.

The condition (7) is necessary and sufficient to make the limit algebra $\mathfrak{g}^0$ well defined. It insures that the *contracting structure constants* $C_{ab}^c(\epsilon)$, defined by

$$\begin{aligned} \sum_{c=1}^N C_{ab}^c(\epsilon)\, X_c^\epsilon &:= [X_a^\epsilon, X_b^\epsilon]_\epsilon = \epsilon^{n_a+n_b}[f_\epsilon(X_a), f_\epsilon(X_b)]_\epsilon = \epsilon^{n_a+n_b} f_\epsilon([X_a, X_b]) \\ &= \epsilon^{n_a+n_b} f_\epsilon\left(\sum_{c=1}^N C_{ab}^c X_c\right) = \sum_{c=1}^N \epsilon^{n_a+n_b-n_c}\, C_{ab}^c X_c^\epsilon, \end{aligned} \tag{8}$$

have finite limits for $\epsilon \to 0$.

The *Inönü-Wigner contraction* is a special case of the above definition, where

$$\begin{aligned} n_i &= 0 \qquad \text{for} \quad i = 1, 2, \ldots, M, \qquad \text{and} \\ n_\alpha &= \text{const.} > 0 \qquad \text{for} \quad \alpha = M+1, M+2, \ldots N\ . \end{aligned} \tag{9}$$

In this case, and by choosing const. = 1 for convenience, we obtain for $\epsilon \to 0$:

$$\begin{aligned}
[X_i^\epsilon, X_j^\epsilon] &= \sum_{k=1}^{M} C_{ij}^k X_k^\epsilon \\
&\Rightarrow \sum_{k=1}^{M} C_{ij}^k X_k^0 \;, \qquad \text{where} \quad X_k^0 := \lim_{\epsilon \to 0} X_k^\epsilon \qquad (10) \\
[X_i^\epsilon, X_\alpha^\epsilon] &= \sum_{k=1}^{M} \epsilon \; C_{i\alpha}^k X_k^\epsilon + \sum_{\beta=M+1}^{N} C_{i\alpha}^\beta X_\beta^\epsilon \\
&\Rightarrow \sum_{\beta=M+1}^{N} C_{i\alpha}^\beta X_\beta^0 \qquad (11) \\
[X_\alpha^\epsilon, X_\beta^\epsilon] &= \sum_{k=1}^{M} \epsilon^2 \; C_{\alpha\beta}^k X_k^\epsilon + \sum_{\gamma=M+1}^{N} \epsilon \; C_{\alpha\beta}^\gamma X_\gamma^\epsilon \\
&\Rightarrow 0 \;. \qquad (12)
\end{aligned}$$

We see that the commutation relations (10) define a subalgebra $\mathfrak{g}_R := \langle X_i^0 \rangle \simeq \langle X_i \rangle$, because C_{ij}^α must vanish to satisfy the condition (7), as was originally concluded in [12]. Note that (12) tells us that $I^0 = \langle X_\alpha^0 \rangle$ is an abelian subalgebra, whereas (11) tells us that I^0 is an ideal of $\mathfrak{g}^0$.

The contractions which are not of the Inönü-Wigner type are called *generalized Inönü-Wigner contractions*. In the present paper we shall encounter one example of Inönü-Wigner contractions and two examples of generalized Inönü-Wigner contractions.

To give the reader an intuitive understanding of the above definitions and notation, let us consider the famous example of contracting the Lorentz algebra to the Galilean algebra: Let e_{ij} denote a basis of 4×4 matrices, defined by $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. They have the following commutation relations

$$[e_{ij}, e_{st}] = \delta_{js} e_{it} - \delta_{it} e_{sj} \;, \quad i, j, s, t = 1, 2, 3, 4. \qquad (13)$$

We define the three contracting boosts by

$$B_i^\epsilon := \epsilon^2 \, e_{i4} + e_{4i} = \epsilon \left(\epsilon \, e_{i4} + \frac{1}{\epsilon} \, e_{4i} \right) =: \epsilon \, f_\epsilon(B_i) \;, \quad i = 1, 2, 3. \qquad (14)$$

These commute as follows

$$[B_i^\epsilon, B_j^\epsilon] = \epsilon^2 \, [e_{i4}, e_{4j}] = \epsilon^2 \, (e_{ij} - e_{ji}) =: -\epsilon^2 \, L_{ij} \Rightarrow 0 \;. \qquad (15)$$

which shows how the the Lorentz algebra $\mathfrak{so}(3,1)$ for $\epsilon = 1$ is contracted to the Galilei algebra, which is the Euclidean algebra $\mathfrak{e}(3)$, in which the limits of the boosts $B_i^0 = e_{4i}$ generate an abelian ideal.

3 The hydrogen algebra $\mathbb{H}_2$ of H_0

Instead of the six generators $\boldsymbol{L}$ and $\boldsymbol{A}$ for the 3-D Kepler problem, only three generators are conserved for the 2-D Kepler problem [10]. These are the third component of angular momentum L_3 and two components of the Runge-Lenz vector $\boldsymbol{A}$:

$$\begin{aligned} L &\equiv L_3 := xp_y - yp_x = p_\varphi \quad \text{and} \\ \boldsymbol{A} &:= (A_1, A_2) = Lp_y\, \hat{\boldsymbol{x}} - Lp_x\, \hat{\boldsymbol{y}} - m\alpha\, \hat{\boldsymbol{r}} \end{aligned} \tag{16}$$

In the following I shall use the following notation:

$$\boxed{h_0 \equiv -2mH_0 \;, \qquad h \equiv -2mH \quad \text{and} \quad \varepsilon \equiv -2mE \;.} \tag{17}$$

For simplicity and also in order to compare my results with those of [10], I shall use from now on *Poisson brackets* instead of commutation relations. But I shall nevertheless refer sometimes to these Poisson brackets as commutators.

The Poisson brackets of the above generators are

$$\begin{aligned} \{L, A_1\} &= A_2 \;, \\ \{A_2, L\} &= A_1 \;, \quad \text{where} \quad h_0 := -2mH_0 \\ \{A_1, A_2\} &= h_0 L \;, \end{aligned} \tag{18}$$

The loop algebra $\mathbb{H}_2$ is spanned by the following generators

$$L^{(2n)} := h_0^n L \quad \text{and} \quad A_i^{(2n+1)} := h_0^n A_i \quad (i = 1, 2) \quad \text{and} \quad n \geq 0 \;. \tag{19}$$

I call the upper index the *grade* of the corresponding operator. According to the above construction, every multiplication by h_0 raises the grade of the generators by 2. With the commutators (18) the set

$$\mathbb{H}_2 := \{A_1^{(2n+1)}, A_2^{(2n+1)}, L^{(2n)} \mid n \geq 0\} \tag{20}$$

becomes a closed Lie algebra, which is a subalgebra of the affine Kac-Moody algebra $A_1^{(1)}$.

3.1 The factor algebra $\mathbb{H}_2\,/I(E)$

The three standard finite-dimensional algebras, $\mathfrak{so}(3), \mathfrak{so}(2,1)$ and $\mathfrak{e}(2)$, can be recovered from $\mathbb{H}_2$, as in [3, 4], as follows: First we define an energy-dependent ideal of $\mathbb{H}_2$ by

$$I(E) := (H_0 - E)\ \mathbb{H}_2\ = (h_0 - \varepsilon)\mathbb{H}_2\,, \qquad \text{where} \quad \varepsilon := -2mE \tag{21}$$

Next, we define the *energy-dependent* factor algebra $\mathbb{H}_2\,/I(E)$ relative to the above ideal. This factor algebra consists of three elements or classes,

$$\mathbb{H}_2\,/I(E) = \{\mathcal{A}_1^\varepsilon,\, \mathcal{A}_2^\varepsilon,\, \mathcal{L}^\varepsilon\}\ , \tag{22}$$

which obey the following commutation relations

$$\boxed{\{\mathcal{L}^\varepsilon, \mathcal{A}_1^\varepsilon\} = \mathcal{A}_2^\varepsilon, \quad \{\mathcal{A}_2^\varepsilon, \mathcal{L}^\varepsilon\} = \mathcal{A}_1^\varepsilon \quad \text{and} \quad \{\mathcal{A}_1^\varepsilon, \mathcal{A}_2^\varepsilon\} = \varepsilon\ \mathcal{L}^\varepsilon\ .} \tag{23}$$

The commutation relations (23) are exactly those of (18), except that the operator h_0 in (18) is now replaced by the numerical parameter ε. This is what physicists usually obtain by *directly* replacing the Hamiltonian H by its energy eigenvalue E.

The above classes can be identified by their representatives, as follows

$$\mathcal{A}_1^\varepsilon = A_1 + I(E)\ , \quad \mathcal{A}_2^\varepsilon = A_2 + I(E)\ , \qquad \text{and} \quad \mathcal{L}^\varepsilon = L + I(E)\ . \tag{24}$$

To see why each *'basic element'* becomes a representative of its class, we recall that quite generally an ideal I of an algebra $\mathfrak{g}$ acts additively as the zero element of the factor algebra $\mathfrak{g}/I$. In our case, this fact yields the following equivalence relation in $\mathbb{H}_2\,/I(E)$,

$$h_0^n X_i \equiv \varepsilon^n X_i \qquad \bmod\ (I(E))\ , \tag{25}$$

where X_i is a basic generator, *i.e.* the element which generates the whole infinite 'tower' $\{h_0^n X_i \mid n = 0, 1, \ldots\}$. The above equivalence relation can be proved easily, as follows

$$h_0^n X_i - \varepsilon^n X_i = (h_0^n - \varepsilon^n) X_i = (h_0 - \varepsilon)\left(\sum_{k=0}^{n-1} \varepsilon^{n-1-k} h_0^k\right) X_i \in I(E)\ . \tag{26}$$

Equation (25) tells us that in the factor algebra we can replace every element $h_0^n X_i \in \mathbb{H}_2$ by $\varepsilon^n X_i$, which is simply a numerical multiple of X_i. Hence, in

$\mathbb{H}_2\,/I(E)$ we can replace every element in the tower $\{h_0^n X_i \mid n = 0, 1, \ldots\}$ by a single element X_i, so that $\mathbb{H}_2\,/I(E)$ is a finite-dimensional algebra generated by the X_i, which in our case are the three elements given in (24).
Note that *the Hamiltonian H_0 by itself is NOT an element of the ideal $I(E)$*.

3.2 Contraction of the factor algebra $\mathbb{H}_2\,/I(E)$

It is easy to check that the map

$$\begin{aligned} f_\varepsilon(\sqrt{\text{ sgn }(\varepsilon)}L_i) &= \frac{1}{\sqrt{|\varepsilon|}}\,\mathcal{A}_i^\varepsilon\ , \quad i = 1, 2\ , \\ f_\varepsilon(L_3) &= \mathcal{L}^\varepsilon\ , \end{aligned} \tag{27}$$

defines an isomorphism between the algebras $\mathfrak{so}(3), \mathfrak{so}(2,1)$ and the factor algebra $\mathbb{H}_2\,/I(E)$ for $\varepsilon < 0,\ \varepsilon > 0$. Hence, by treating $\varepsilon = -2mE$ as a contraction parameter ϵ, the classes $\mathcal{A}_1^\varepsilon, \mathcal{A}_2^\varepsilon$ and $\mathcal{L}^\varepsilon$ with the commutation relations (23) can be regarded as the generators of a *contracting* algebra $\mathfrak{g}^\epsilon$ (see Sec. 2), for $\varepsilon \neq 0$ (!).

For $\varepsilon \to 0$ the algebras $\mathbb{H}_2\,/I(\varepsilon)$ are contracted to $\mathbb{H}_2\,/I(0)$, whose commutation relations follow from (23)

$$\{\mathcal{L}^0, \mathcal{A}_1^0\} = \mathcal{A}_2^0\ , \quad \{\mathcal{A}_2^0, \mathcal{L}^0\} = \mathcal{A}_1^0\ , \quad \{\mathcal{A}_1^0, \mathcal{A}_2^0\} = 0\ , \tag{28}$$

Since these are the commutation relations of the Euclidean algebra $\mathfrak{e}(2)$, it follows that $\mathbb{H}_2\,/I(0) \simeq \mathfrak{e}(2)$. Since f_ϵ is an isomorphism for $\varepsilon \neq 0$, we conclude that a contraction of $\mathbb{H}_2\,/I(\varepsilon)$ for the non-broken Hamiltonian H_0 in (5) is the same as the well-known contraction of $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ to the Euclidean algebra $\mathfrak{e}(2)$, as $\varepsilon \to 0$. Note that the number of generators remains the same after contraction. In the present case, the contraction is of the Inönü-Wigner type.

In the next two sections we shall see that the Factor algebras associated with the 'broken Hamiltonian' H of (4) yield two contractions of the generalized Inönü-Wigner type.

4 The loop algebra $\mathfrak{L}_1(\beta)$ of H in (4)

For the Hamiltonian (4) there exist a generalized conserved Runge-Lenz vector [10], which is given by

$$\mathbf{M} \ \equiv\ \mathbf{M}(\beta) := \boldsymbol{A} - m\beta\sqrt{r}\sin(\varphi/2)\ \hat{\boldsymbol{\phi}}(\varphi)$$

$$= \left(\frac{p_\varphi^2}{r} - m\beta\right)\ \hat{\boldsymbol{r}}(\varphi) - \left(p_r p_\varphi \ + \ m\beta\sqrt{r}\sin(\varphi/2)\right)\ \hat{\boldsymbol{\phi}}(\varphi)\ . \quad (29)$$

Note that $\mathbf{M}(0) = \boldsymbol{A}$. The commutator of the two components of $\mathbf{M}$ in (29) yield a third conserved quantity, which I shall denote by S (It is called $-I$ in [10]) : It is defined by [10]

$$S := \{M_1, M_2\} = h\ p_\varphi - m\beta(p_r r^{1/2}\sin(\varphi/2) + p_\varphi r^{-1/2}\cos(\varphi/2))\ , \quad (30)$$

The commutators of S with M_i are [10]

$$\{S, M_1\} = h\ M_2 \quad \text{and} \quad N_1 := \{M_2, S\} = hM_1 - m^2\beta^2/2\ . \quad (31)$$

We can summarize the above commutators, as follows

$$\{S, N_1\} = h^2 M_2\ , \quad \{M_2, S\} = N_1 \quad \text{and} \quad \{N_1, M_2\} = hS\ . \quad (32)$$

Therefore, I call the following three generators, *'basic generators'*

$$N_1, \quad M_2, \quad \text{and} \quad S, \quad (33)$$

because they can yield a closed algebra by multiplying them with powers of h as in (34) below. *The above basic generators were chosen, such that none of them vanishes nor blows up for* $H = 0$.

As before, since H commutes with the basic generators, we can close the algebra in (32) by including the following generators

$$h^n M_2\ , \quad h^n N_1\ , \quad h^n S\ , \quad n \geq 0\ . \quad (34)$$

It is interesting to note that by commuting the basis generators N_1, M_2 and S, among themselves and with their commutators, we can never produce hM_2. This means that it is possible to obtain a closed algebra even without hM_2. Nevertheless, I included hM_2 in (34) in order to obtain a closed algebra which is generated by the basic generators over the polynomial ring $R[h]$.

A crucial step in identifying the algebra generated by the operators in (34) is to *assign grades* to each operator, because for Lie algebras of the Kac-Moody type the sum of the grades (which I am writing as upper indices) must be conserved under commutation. It is easy to check, that the following identification of the grades is consistent

$$\boxed{M_2^{(2n+1)} := h^n M_2\ , \quad N_1^{(2n+3)} := h^n N_1\ , \quad S^{(2n+2)} := h^n S\ , \quad n \geq 0\ .} \quad (35)$$

For example, using (32) we obtain

$$\{N_1^{(2m+3)}, M_2^{(2n+1)}\} = h^{m+n}\{N_1, M_2\} = h^{m+n+1}S = S^{(2m+2n+4)} \ .$$

Therefore the above infinite generators span the following graded Loop algebra of the Kac-Moody type,

$$\mathfrak{L}_1(\beta) := \{M_2^{(2n+1)}, N_1^{(2n+3)}, S^{(2n+2)} \mid n \geq 0\} \ . \tag{36}$$

Note that the basic generators are graded, as follows

$$M_2 = M_2^{(1)} \ , \qquad S = S^{(2)} \quad \text{and} \quad N_1 = N_1^{(3)} \ .$$

4.1 The factor algebra $\mathfrak{L}_1(\beta)/I_1(E,\beta)$

As before, the factor algebra

$$\mathfrak{L}_1(\beta)/I_1(E,\beta) = \{\mathcal{M}_2^\varepsilon, \mathcal{N}_1^\varepsilon, \mathcal{S}^\varepsilon\}, \tag{37}$$

relative to the following energy-dependent ideal

$$I_1(E,\beta) := (H-E)\ \mathfrak{L}_1(\beta) = (h-\varepsilon)\mathfrak{L}_1(\beta) \ , \qquad \text{where} \quad \varepsilon = -2mE \ . \tag{38}$$

has three classes, which commute as follows

$$\boxed{\{\mathcal{S}^\varepsilon, \mathcal{N}_1^\varepsilon\} = \varepsilon^2 \mathcal{M}_2^\varepsilon, \quad \{\mathcal{M}_2^\varepsilon, \mathcal{S}^\varepsilon\} = \mathcal{N}_1^\varepsilon \quad \text{and} \quad \{\mathcal{N}_1^\varepsilon, \mathcal{M}_2^\varepsilon\} = \varepsilon\ \mathcal{S}^\varepsilon \ .} \tag{39}$$

4.2 Contraction of the factor algebra $\mathfrak{L}_1/I_1(E,\beta)$

In the present case we need a different map

$$\begin{array}{rcl} f_\varepsilon(\sqrt{\ \text{sgn}\ (\varepsilon)}L_1) & = & \mathcal{N}_1^\varepsilon/|\varepsilon|^{3/2} =: \widehat{\mathcal{N}_1}^\varepsilon \\ f_\varepsilon(\sqrt{\ \text{sgn}\ (\varepsilon)}L_2) & = & \mathcal{M}_2^\varepsilon/|\varepsilon|^{1/2} =: \widehat{\mathcal{M}_2}^\varepsilon \\ f_\varepsilon(L_3) & = & \mathcal{S}^\varepsilon/|\varepsilon| =: \widehat{\mathcal{S}}^\varepsilon \ , \end{array} \tag{40}$$

which again defines an isomorphism between the algebras $\mathfrak{so}(3), \mathfrak{so}(2,1)$ and the factor algebra $\mathfrak{L}_1/I_1(E,\beta)$ for $\varepsilon < 0$, $\varepsilon > 0$. The three generators $\widehat{\mathcal{N}_1}^\varepsilon, \widehat{\mathcal{M}_2}^\varepsilon$ and $\widehat{\mathcal{S}}^\varepsilon$ may be called 'normalized' generators.

The contraction of the factor algebras $\mathfrak{L}_1/I_1(E,\beta)$ yields $\mathfrak{L}_1/I_1(0,\beta)$, whose commutation relations follow from (39). They are given by

$$\begin{aligned}
[\mathcal{N}_1^0, \mathcal{M}_2^0] &= \lim_{\varepsilon\to 0} \varepsilon^{3/2+1/2-1} \mathcal{S}^\varepsilon = 0 \\
[\mathcal{S}^0, \mathcal{N}_1^0] &= \lim_{\varepsilon\to 0} \varepsilon^{1+3/2-1/2} \mathcal{M}_2^\varepsilon = 0 \qquad (41) \\
[\mathcal{M}_2^0, \mathcal{S}^0] &= \lim_{\varepsilon\to 0} \varepsilon^{1/2+1-3/2} \mathcal{N}_1^\varepsilon = \mathcal{N}_1^0 \; . \qquad (42)
\end{aligned}$$

These are the commutation relations of the Heisenberg-Weyl algebra $\mathfrak{h}_3 = \mathfrak{w}_1$, as we can see by using the following map

$$\mathcal{M}_2^0 \to \partial_x, \qquad \mathcal{S}^0 \to x, \qquad \text{and} \quad \mathcal{N}_1^0 \to 1 \; .$$

It is important to note that in the factor algebra $\mathfrak{L}_1(\beta)/I_1(E,\beta)$ we are NOT allowed to replace the h in N_1 of (31) by ε, since neither h nor $(h-\varepsilon)M_1$ are elements of the ideal $I_1(E,\beta)$. Hence, N_1 is independent of E and thus it should NOT be replaced by the constant $-m^2\beta^2/2$ for $E=0$.

5 A second loop algebra $\mathfrak{L}_2(\beta)$ of H in (4)

In this section I show that a different choice of the basic generators yields different contractions. Instead of the three generators in (33) I now choose the basic generators, as follows

$$N_1, \qquad N_2 := hM_2 \quad \text{and} \quad S \; . \qquad (43)$$

The choice of N_2 in (43) may seem unjustified. But I chose it nevertheless in order to illustrate how we can obtain different contractions by simply removing some generators from the *same* loop algebra.

The choice (43) would seem less strange, had I kept the phase angle γ in (4) arbitrary : In this case I would have obtained

$$\begin{aligned}
\tilde{N}_1 &:= \{\tilde{M}_2, \tilde{S}\} = \tilde{h}\tilde{M}_1 - \frac{1}{2}m^2\beta^2\cos\gamma \quad \text{and} \\
\tilde{N}_2 &:= \{\tilde{S}, \tilde{M}_1\} = \tilde{h}\tilde{M}_2 - \frac{1}{2}m^2\beta^2\sin\gamma \qquad (44)
\end{aligned}$$

where the tilde over the quantities denote the quantities of the previous section, but with $\gamma \neq 0$. Hence, for γ arbitrary, the $\tilde{N}_1, \tilde{N}_2$ and $\tilde{S}$ would

have seemed to be the natural choice for the basic generators. In fact, these generators were originally chosen by Leach and Flessas [10, Eq. (3.4.5)] as the symmetry generators of the Hamiltonian (4) for $E \neq 0$. However, for $E = 0$ they made a different choice, and chose the following linear combinations of $\tilde{N}_1$ and $\tilde{N}_2$

$$\begin{aligned} N_1 &= \cos\gamma\tilde{N}_1 + \sin\gamma\tilde{N}_2 = hM_1 - \frac{1}{2}m^2\beta^2 \\ M_2 &= \frac{1}{h}(\sin\gamma\tilde{N}_1 - \cos\gamma\tilde{N}_2) \ . \end{aligned} \tag{45}$$

We see that their second choice (45) corresponds exactly to the generators which I used in Sec. 4, by setting $\gamma = 0$ from the beginning. This explains why they were able to obtain the algebra $\mathfrak{h}_3 = \mathfrak{w}_1$ as the symmetry algebra for $E = 0$; for $E \neq 0$ it does not matter which linear combinations one chooses: one always obtain $\mathfrak{so}(3)$ or $\mathfrak{so}(2,1)$.

The generators in (43) commute, as follows

$$\{N_1, N_2\} = h^2 S, \quad \{N_2, S\} = hN_1, \quad \text{and} \quad \{S, N_1\} = h\ N_2 \tag{46}$$

Following the same procedure as before, the following operators

$$\begin{aligned} N_1^{(2n+3)} &:= h^n N_1\ , \\ N_2^{(2n+3)} &:= h^n N_2\ , \qquad \text{for} \quad n \geq 0\ , \\ S^{(2n+2)} &:= h^n S\ , \end{aligned} \tag{47}$$

yield the following Loop algebra, provided one uses the grading in (47)

$$\mathfrak{L}_2 := \{N_1^{(2n+3)}, N_2^{(2n+3)}, S^{(2n+2)} \mid n \geq 0\} \tag{48}$$

5.1 The factor algebra $\mathfrak{L}_2(\beta)/I_2(E,\beta)$

The factor algebra in this case consists also of three classes, namely

$$\mathfrak{L}_2(\beta)/I_2(E,\beta) = \{\mathcal{N}_1^\varepsilon, \mathcal{N}_2^\varepsilon, \mathcal{S}^\varepsilon\}\ , \tag{49}$$

where

$$I_2(E,\beta) := (H - E)\mathfrak{L}_2(\beta) = (h - \varepsilon)\mathfrak{L}_2(\beta)\ . \tag{50}$$

These classes commute as follows

$$\boxed{\{\mathcal{N}_1^\varepsilon, \mathcal{N}_2^\varepsilon\} = \varepsilon^2\ \mathcal{S}^\varepsilon\ , \quad \{\mathcal{N}_2^\varepsilon, \mathcal{S}^\varepsilon\} = \varepsilon\mathcal{N}_1^\varepsilon \quad \text{and} \quad \{\mathcal{S}^\varepsilon, \mathcal{N}_1^\varepsilon\} = \varepsilon\mathcal{N}_2^\varepsilon\ .} \tag{51}$$

Hence, in this case we obtain for $\varepsilon \to 0$ a contraction of $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ to an abelian algebra, which I denote by R^3. This is a generalized Inönü-Wigner contraction.

Note that if $N_2 = hM_2$, as defined in (43), then $N_2^{(2n+3)} = M_2^{(2n+3)}$, so that $\mathfrak{L}_2$ is just a subalgebra of $\mathfrak{L}_1$, with just the element M_2 removed, *i.e.*

$$\mathfrak{L}_2 = \mathfrak{L}_1 \backslash M_2^{(1)} = \mathfrak{L}_1 \backslash M_2 \ . \tag{52}$$

Again note that in the factor algebra $\mathfrak{L}_2(\beta)/I_2(E,\beta)$ we are NOT allowed to replace $N_2 = hM_2$ by εM_2, since $(h-\varepsilon)M_2$ is NOT an element of the ideal $I_2(E,\beta)$, because (52) tells us that $M_2 \notin \mathfrak{L}_2$. Thus, the class $\mathcal{N}_2^0 = N_2 + I_2(0) \neq I_2(0)$, which means that the contracted factor algebra $\mathfrak{L}_2(\beta)/I_2(0,\beta)$ remains three-dimensional, as it should. Note that with the formal factor-algebra construction every one of the three generators is well defined and will not vanish in the limit $\varepsilon \to 0$, so that this realization is *saved* [15]. *In contrast, if instead we follow the standard procedure and work directly with the generators* N_1, N_2 *and* S *and just replace the* h *by* ε*, then* $N_2 = hM_2$ *will become* $N_2 = \varepsilon M_2$ *and thus it will vanish in the limit* $\varepsilon \to 0$, so that N_2 will not be saved.

6 Summary and conclusions

In the present paper I constructed two Kac-Moody loop algebras $\mathfrak{L}_1(\beta)$ and $\mathfrak{L}_2(\beta)$. The second algebra $\mathfrak{L}_2(\beta)$ was studied simply to show that one has the freedom of constructing more than one loop algebra from the conserved constants of motion, M_1, M_2, S and H. These two infinite-dimensional algebras are operator-valued and thus do NOT depend on energy E.

To study contractions I first constructed E-dependent factor algebras, in order to obtain finite-dimensional algebras out of the infinite-dimensional ones. As I explained in Eq. (25), this construction enables us to replace all the higher generations $X_i^n := h^n X_i$ by $\varepsilon^n X_i$, so that within the factor algebras all the generators X_i^n become numerical multiples of the basic generators $X_i = X_i^0$. In particular, for $E = 0$ we obtain $\varepsilon^n X_i = 0$ for $n \geq 1$.

To avoid any misunderstanding, I want to emphasize again that I am NOT contracting the infinite-dimensional Kac-Moody loop algebras, $\mathbb{H}_2, \mathfrak{L}_1(\beta)$ and $\mathfrak{L}_2(\beta)$; **I am only contracting their (3-dimensional) factor algebras**, $\mathbb{H}_2/I(E), \mathfrak{L}_1(\beta)/I_1(E,\beta)$ and $\mathfrak{L}_2(\beta)/I_2(E,\beta)$, by using the energy E as the contraction parameter. It is interesting that although all the three factor

algebras are isomorphic to $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ for $E < 0$ and $E > 0$, they contract for $E \to 0$ to three different algebras $\mathfrak{e}(2), \mathfrak{h}_3 = \mathfrak{w}_1$ and R^3, which are also 3-dimensional. The first contraction is of the Inönü-Wigner type while the other two are of the generalized Inönü-Wigner type. In all these contractions the dimension of the algebras is preserved, since the factor algebras do not change their dimensions as $E \to 0$. These contractions are summarized in table 1.

The effect of symmetry breaking $H(\beta)$ manifests itself differently in in the

Hamiltonian	Factor algebra	$E < 0$	$E = 0$	$E < 0$
H_0 in (5)	$\mathbb{H}_2 / I(E)$	$\mathfrak{so}(3)$	$\mathfrak{e}(2)$	$\mathfrak{so}(2,1)$
H in (4)	$\mathfrak{L}_1(\beta)/I_1(E,\beta)$	$\mathfrak{so}(3)$	$\mathfrak{h}_3 = \mathfrak{w}_1$	$\mathfrak{so}(2,1)$
H in (4)	$\mathfrak{L}_2(\beta)/I_2(E,\beta)$	$\mathfrak{so}(3)$	R^3	$\mathfrak{so}(2,1)$

Table 1: The three factor algebras $\mathbb{H}_2 / I(E)$ and $\mathfrak{L}_i/I_i$ of the loop algebras $\mathbb{H}_2$ and $\mathfrak{L}_i$ relative to the corresponding energy-dependent ideals $I(E)$ and $I_i(E,\beta)$. For $E \neq 0$ all three factor algebras are isomorphic to $\mathfrak{so}(3)$ for $E < 0$ and to $\mathfrak{so}(2,1)$ for $E > 0$, but yield *different contractions* for $E \to 0$.

standard and the Kac-Moody treatments: In the standard procedure, which was followed by Leach *et. al.* [9, 10], the symmetry algebras for H_0 and $H(\beta)$ are exactly the same, namely $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$. The effect of symmetry breaking manifests itself only for $E \neq 0$.

In contrast, as we shall now see, the Loop algebras $\mathfrak{L}_1$ and $\mathfrak{L}_2$ for the 'broken Hamiltonian' $H(\beta)$ are *smaller* than the hydrogen algebra $\mathbb{H}_2$ for H_0 (irrespective of the energy!). They are smaller by two and three elements, respectively, thereby revealing the symmetry breaking:

To compare $\mathfrak{L}_1$ and $\mathfrak{L}_2$ with $\mathbb{H}_2$, I define two maps, as follows: $f_1 : \mathfrak{L}_1 \mapsto \mathbb{H}_2$, defined by

$$f_1(N_1^{(2n+3)}(\beta)) \ := \ A_1^{(2n+3)} \ ,$$

$$
\begin{array}{rcll}
f_1(M_2^{(2n+1)}(\beta)) & := & A_2^{(2n+1)} \ , & \qquad \text{for} \quad n \geq 0 \ , \\
f_1(S^{(2n+2)}(\beta)) & := & L^{(2n+2)} \ , &
\end{array}
\tag{53}
$$

and $f_2 : \mathfrak{L}_2 \mapsto \mathbb{H}_2$, defined by

$$
\begin{array}{rcll}
f_2(N_1^{(2n+3)}(\beta)) & := & A_1^{(2n+3)} \ , & \\
f_2(N_2^{(2n+3)}(\beta)) & := & A_2^{(2n+3)} \ , & \qquad \text{for} \quad n \geq 0 \ , \\
f_2(S^{(2n+2)}(\beta)) & := & L^{(2n+2)} \ , &
\end{array}
\tag{54}
$$

It is easy to check that these two maps, which keep the grades of the generators unchanged, define isomorphisms from $\mathfrak{L}_1$ and $\mathfrak{L}_2$ onto subalgebras of $\mathbb{H}_2$ of codimension 2 and 3, respectively. Hence,

$$
\mathbb{H}_2 \ = \left\{ \begin{array}{ll} f_1(\mathfrak{L}_1(\beta)) \cup \{L, A_1\} \ , & \qquad \text{and} \\ & \\ f_2(\mathfrak{L}_2(\beta)) \cup \{L, A_1, A_2\} \ . & \end{array} \right.
\tag{55}
$$

Thus, we can conclude that symmetry breaking of the type (4) reduces the loop algebra $\mathbb{H}_2$ of the original system H_0 by only finite number of generators. By constructing the corresponding factor algebras, I obtained different contractions depending on the missing terms (see Table 1).

By defining the ε-dependent ideals and constructing the factor algebras, we are essentially replacing each infinite-dimensional *'tower'* $\{X_i^n\}$ by one element $X_i^{n_i^{min}}$ which has the *lowest grade. By removing generators from the original loop algebra, we increase the grade of the corresponding basic generators.* This in turn increases the powers of the contraction parameter ε which multiply the structure constants of the original algebra $\mathfrak{g}$, which is being contracted.

The results obtained in the present paper suggest a *general procedure for defining contractions via Kac-Moody formalism*, as follows:

- Start with of a finite dimensional Lie algebra $\mathfrak{g}$, which may be graded, via s-dimensional automorphism, as follows

$$
\mathfrak{g} = \bigoplus_{k=0}^{s-1} \mathfrak{g}_k \ , \qquad \text{with} \quad [\mathfrak{g}_i, \, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j} \ ,
\tag{56}
$$

  where the indices are modulo s .

- Then consider the *positive* subalgebra of a general (twisted or untwisted) loop algebra of a finite dimensional algebra $\mathfrak{g}$,

$$\mathfrak{L} = \left\{ \bigoplus_{k=0}^{s-1} z^{sn+k} \otimes \mathfrak{g}_k \;\mid\; n \geq 0 \right\} \;, \tag{57}$$

  where t may be a scalar or an operator which commutes with all the generators of $X_i \in \mathfrak{g}$.

- Then remove some generators from $\mathfrak{L}$, and make sure that the *remaining set* $\mathfrak{L}_R$ yields a subalgebra of $\mathfrak{L}$. This is not automatic: see, for example, the conditions in (60) below. Then make sure that the set

$$I_R(\varepsilon) = (z - \varepsilon)\mathfrak{L}_R \tag{58}$$

  is an ideal of $\mathfrak{L}_R$, since for some choices $(z - \varepsilon)\mathfrak{L}_R$ is not a subalgebra of $\mathfrak{L}_R$.

- Finally, define the factor algebras $\mathfrak{L}_R/I_R(\varepsilon)$, which will be isomorphic to one or two real forms of $\mathfrak{g}$, depending on the sign of parameter ε. The ε can be used as a contraction parameter. One may get different contractions for the same original algebra $\mathfrak{g}$ as $\varepsilon \to 0$, depending on the removed generators.

For example, we can define subalgebras of $\mathbb{H}_2$ by

$$\mathbb{H}_2(n_1, n_2, n_3) := \langle h^{n_1+n} A_1, h^{n_2+n} A_2, h^{n_3+n} L_3, n \geq 0 \rangle \tag{59}$$

if the n_i satisfy the following conditions

$$n_1 + n_2 - n_3 + 1 \geq 0 \;, \quad n_3 + n_1 - n_2 \geq 0, \quad n_3 + n_2 - n_1 \geq 0 \;. \tag{60}$$

In particular, as I showed explicitly in (53) and (54), the loop algebras $\mathfrak{L}_i$ are isomorphic to the following subalgebras of $\mathbb{H}_2$, and thus give us intuitive physical realizations of the formal definition in (59):

$$\mathfrak{L}_1 \simeq \mathbb{H}_2(1,0,1) \quad \text{and} \quad \mathfrak{L}_2 \simeq \mathbb{H}_2(1,1,1) \tag{61}$$

In these subalgebras of $\mathbb{H}_2$ the conditions (60) are clearly satisfied.

The conditions (60) follow from two different arguments:

1. The generators of the subalgebra $\mathbb{H}_2(n_1, n_2, n_3)$ commute, as follows

$$\begin{array}{lcll} [h^{n_1}A_1, h^{n_2}A_2] & = & h^{n_1+n_2+1}L_3, & \text{hence } n_3 \le n_1+n_2+1 \\ [h^{n_3}L_3, h^{n_1}A_1] & = & h^{n_3+n_1}A_2, & \text{hence } n_2 \le n_3+n_1 \\ [h^{n_3}L_3, h^{n_2}A_2] & = & -h^{n_3+n_2}A_1, & \text{hence } n_1 \le n_3+n_2 \end{array} \qquad (62)$$

   The conditions (60) are necessary to ensure that the r.h.s. of the above commutators are elements of $\mathbb{H}_2(n_1, n_2, n_3)$.

2. In the factor algebra $\mathbb{H}_2(n_1, n_2, n_3)/(\ (h-\varepsilon)\mathbb{H}_2(n_1, n_2, n_3)\ )$ only the generators with lowest grade are linearly independent. Their commutators are

$$\begin{array}{lcl} [\epsilon^{n_1}A_1, \epsilon^{n_2}A_2] & = & \epsilon^{n_1+n_2-n_3+1}(\epsilon^{n_3}L_3) \\ [\epsilon^{n_3}L_3, \epsilon^{n_1}A_1] & = & \epsilon^{n_3+n_1-n_2}(\epsilon^{n_2}A_2) \\ [\epsilon^{n_3}L_3, \epsilon^{n_2}A_2] & = & -\epsilon^{n_3+n_2-n_1}(\epsilon^{n_1}A_1) \end{array} \qquad (63)$$

   Hence, in order for the r.h.s. of the above three equations to exist as $\epsilon \to 0$, the exponents of ϵ must be non-negative. This requirement yields exactly the same conditions on the n_i as those given in (62), which were necessary for the existence of subalgebras of $\mathbb{H}_2$.

- More generally, given an N-dimensional semisimple algebra $\mathfrak{g}$, we can define subalgebras $\mathfrak{g}_{\mathbf{n}}$ by

$$\mathfrak{g}_{\mathbf{n}} := \langle h^{n_i}X_i \mid n \ge 0 \quad \text{and} \quad i = 1, 2, \ldots N\rangle \ . \qquad (64)$$

  Instead of an operators h, with $[h, X_i] = 0$, we can also use a formal variable z.

  These subalgebras yield well-defined contractions via the factor-algebra $\mathfrak{g}_{\mathbf{n}}/(\ (h-\epsilon)\mathfrak{g}_{\mathbf{n}}\ )$, provided the n_i satisfy the general condition (7), namely $\epsilon^{n_i+n_j-n_k}C_{ij}^k < \infty$.

Finally, we note that if we *first* take the limit $\beta \to 0$ in the *'deformed Hamiltonian'* $H(\beta)$ of (4) we recover the original Hamiltonian H_0 and thus obtain the symmetry algebra $\mathbb{H}_2$ and consequently the contraction to $\mathfrak{e}_2$. In contrast, if we construct the loop symmetry algebras $\mathfrak{L}_i(\beta)$ for $\beta \ne 0$ first, then the $\mathfrak{L}_i(\beta)$ (and also their factor algebras) remain unchanged as we let

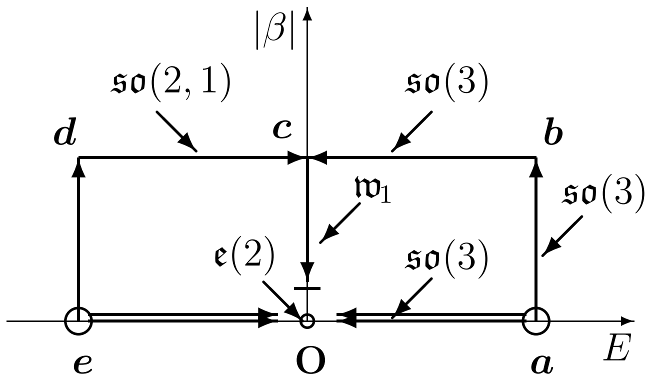

Figure 1: The figure illustrates the '*DC hysteresis*' in the (E, β) parameter plane, by comparing the contraction limits $E \to 0$ of the factor algebras $\mathbb{H}_2/I(E)$ and $\mathfrak{L}_1/I_1(E,\beta)$: If we contract the factor algebras $\mathbb{H}_2/I(E)$ of H_0 along the horizontal energy E-axis, which corresponds to $\beta = 0$, we obtain $\mathfrak{e}(2)$. This contraction is indicated by the double arrows ($\boldsymbol{e} \Rightarrow \mathbf{O} \Leftarrow \boldsymbol{a}$). In contrast, for $\beta \neq 0$ the contraction of $\mathfrak{L}_1/I_1(E,\beta)$ of H yields the Weyl algebra $\mathfrak{L}_1/I_1(0,\beta) = \mathfrak{h}_3 = \mathfrak{w}_1$, as illustrated by $\boldsymbol{d} \to \boldsymbol{c} \leftarrow \boldsymbol{b}$. Finally, taking the limit of $\mathfrak{L}_1/I_1(0,\beta)$ as $\beta \to 0$ downwards along the vertical $|\beta|$-axis to the origin $(E,\beta) = (0,0)$ leaves the algebra $\mathfrak{h}_3 = \mathfrak{w}_1$ unchanged. Thus, the two paths originating in $\boldsymbol{a}$ yields different limits: $\mathfrak{so}(3) \simeq \mathbb{H}_2/I(E) \simeq \mathfrak{L}_1(\beta)/I_1(E,\beta) \to \mathfrak{L}_1(\beta)/I_1(0,\beta) \simeq \mathfrak{L}_1(0)/I_1(0,0) \simeq \mathfrak{h}_3 = \mathfrak{w}_1 \neq \mathfrak{e}(2) \simeq \mathbb{H}_2/I(0) \Leftarrow \mathbb{H}_2/I(E)$.

$\beta \to 0$, and thus we do NOT go back to $\mathbb{H}_2$ (and its factor algebras). I call this phenomenon the **DC (deformation-contraction) hysteresis**, since *we obtain different contractions depending on the order of taking the limits $E \to 0$ and $\beta \to 0$.* The subtlety of the DC hysteresis, which yields $\mathfrak{h}_3 = \mathfrak{w}_1$ instead of $\mathfrak{e}_2$ is illustrated in Fig. 1.

Acknowledgments

I am happy to thank my daughter, Claudia Daboul, for reading the manuscript and making useful comments. It is also a pleasure to thank the members of CCF, and in particular Bernardo Wolf, for their hospitality.

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